

Minimality, transitivity and sensitivity of non-uniform cellular automata.

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Abstract. Every transitive cellular automaton (CA) is sensitive to initial conditions. We study this implication in the more general context of non-uniform cellular automata (NUCA) with finitely many different local update rules assigned to cells. We introduce an odometer NUCA on $\{0, 1, 2\}^{\mathbb{N}}$ which is nearly uniform in the sense that only the first cell uses a different local rule. This can be used to construct a two-dimensional NUCA that is minimal – and hence transitive – but that is not sensitive to initial conditions. Additionally we find that if the assignment of local rules in the cells is recurrent then transitivity implies sensitivity.

Keywords: Non-uniform cellular automata, odometer, adding machine, minimality, sensitivity, transitivity.

1 Introduction

Classical cellular automata (CA) are homogeneous both in space and time. All cells use the same local update rule, and the same rule is used at all discrete time steps. A non-uniform cellular automaton (NUCA) is a generalization where different cells may use different update rules. However, in our formalism, there are only finitely many different rules available, and the assignment of rules – called the rule distribution – does not change over time. In each cell the NUCA applies the local rule of that cell, given by the rule distribution. Non-uniformity greatly changes possible behaviors of the system [2, 4, 3].

In this work we investigate the relationship of transitivity and sensitivity in NUCA. If a uniform cellular automaton is transitive then it is also sensitive [1]. We introduce a two-dimensional NUCA that is minimal – and hence transitive – but equicontinuous – and hence not sensitive. In the construction we use a one-way cellular automaton with an odometer behavior. A detailed version of the article containing the full proofs will appear in Lecture Notes in Computer Science [5].

2 A three state odometer on $\mathbb{N}$

We consider a NUCA over a subset $X \subseteq \mathbb{Z}^d$. Let $\Sigma = \{0, 1, 2\}$ and $X = \mathbb{N}$. Let f be the local rule that uses the neighborhood $\{-1, 0\}$ and the transition

function that (i) keeps the state unchanged if the left neighbor is in state 0, (ii) swaps states $0 \leftrightarrow 1$ if the left neighbor is in state 1, and (iii) swaps states $0 \leftrightarrow 2$ if the left neighbor is in state 2, and let g be the local rule with the neighborhood $\{0\}$ and the transition function that makes the states cycle $0 \mapsto 1 \mapsto 2 \mapsto 0 \mapsto \dots$. Let $\theta \in \{f, g\}^{\mathbb{N}}$ be the local rule distribution $g f f f \dots$ on $\mathbb{N}$. Figure 1 shows the beginning of the space-time diagram starting from the 0-uniform initial configuration. We call the global transition function $H = H_\theta : \{0, 1, 2\}^{\mathbb{N}} \rightarrow \{0, 1, 2\}^{\mathbb{N}}$ the *three state odometer*.

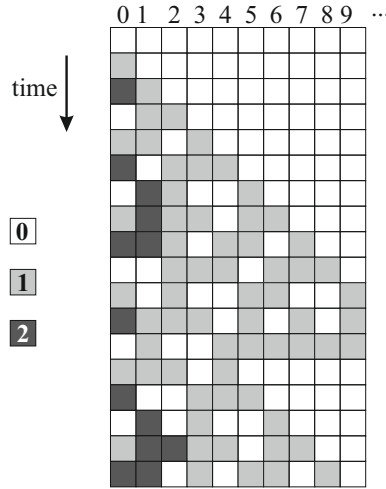


Fig. 1. The evolution of the first ten cells for 18 time steps in the three state odometer. Initially all cells are in state 0.

The following can be shown about the three state odometer.

Theorem 1. *The three state odometer is minimal and not sensitive.*

The three state odometer can also be embedded into $\mathbb{Z}^2$ in a spiral centered on the origin, giving a 2-dimensional NUCA that is minimal but not sensitive.

Theorem 2. *There exists a 2-dimensional NUCA that is minimal – and hence also transitive – but not sensitive.*

Additionally, the property that transitivity implies sensitivity in uniform CA can be shown about NUCA with recurrent rule distributions.

Theorem 3. *Let $\theta \in \mathcal{R}^{\mathbb{Z}^d}$ be a recurrent rule distribution such that $H_\theta : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ is transitive. Then H_θ is sensitive.*

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